ON THE (DUAL) BLASCHKE DIAGRAM

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ABSTRACT. In this paper we study the behavior of the roots of the Steiner polynomial of a convex body when we embed it in a higher dimensional space. In the 3-dimensional case, the involved sets will follow a precise pattern when they are mapped into the Blaschke diagram. We also construct and characterize the so-called dual Blaschke diagram. As an immediate consequence of it we will get a new characterization of dual quermassintegrals in dimension $n = 3$.

1. Introduction and notation

Let $\mathcal{K}^n$ denote the set of convex bodies in $\mathbb{R}^n$, i.e., the family of all non-empty convex and compact subsets $K \subset \mathbb{R}^n$, and with $\mathcal{K}_0^n$ we represent the family of those convex bodies containing the origin in their interior. In particular, we write B_n to denote the Euclidean unit ball. The volume of a measurable set $M \subset \mathbb{R}^n$, i.e., its n-dimensional Lebesgue measure is denoted by $\mathrm{vol}(M)$, or by $\mathrm{vol}_n(M)$ if the distinction of the dimension is needed.

For a convex body $K \in \mathcal{K}^n$ and a non-negative real number λ, the volume of the Minkowski sum $K + \lambda B_n$ is a polynomial of degree n in λ, and it is written as

$$\mathrm{vol}(K + \lambda B_n) = \sum_{i=0}^{n} \binom{n}{i} \mathrm{W}_i(K)\lambda^i \tag{1.1}$$

(see [11]). This expression is called the *Steiner formula* of K. The coefficients $\mathrm{W}_i(K)$ are the *quermassintegrals* of K, and they are a special case of the more general defined *mixed volumes*, for which we refer to [10, Section 5.1]. In particular, $\mathrm{W}_0(K) = \mathrm{vol}(K)$, $\mathrm{W}_n(K) = \mathrm{vol}(B_n) =: \kappa_n$, $n\mathrm{W}_1(K)$ is the surface area of K and $(2/\kappa_n)\mathrm{W}_{n-1}(K)$ is the so-called *mean width* of K ([10, page 50]). Quermassintegrals do also satisfy a system of Steiner type formulae (see [10, Note 6 for Section 4.2]), namely,

$$\mathrm{W}_i(K + \lambda B_n) = \sum_{k=0}^{n-i} \binom{n-i}{k} \mathrm{W}_{i+k}(K)\lambda^k. \tag{1.2}$$

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In several recent articles (see [4] and the references therein) properties of the roots of the Steiner polynomial $f_{K,B_n}(z) := \sum_{i=0}^{n}\binom{n}{i}\mathrm{W}_i(K)z^i$ (regarded as a polynomial in a complex variable $z \in \mathbb{C}$, cf. (1.1)), but in the more general setting of the Minkowski Geometry, are investigated: structural properties of the cone of roots, convexity, topology, monotonicity, stability...

In [6] the 3-dimensional (classical) Steiner polynomial was considered: convex bodies were classified in terms of relations (equations and inequalities) among the quermassintegrals, with respect to the algebraic type of the roots of the polynomial; moreover, this kind of classification turned out to be closely related to a well-known open question in Convexity, namely, the so-called Blaschke problem.

This paper can be divided into two parts: on one hand we will continue the investigation of the above mentioned relation between properties of the roots of Steiner polynomials and the Blaschke diagram; on the other hand, in the last section of the article, we will study the corresponding Blaschke problem in the setting of the so-called dual Brunn-Minkowski theory.

We conclude the introduction with a brief review on the Blaschke problem.

1.1. **The Blaschke problem.** In [2] Blaschke asked for a characterization of the set of all points in $\mathbb{R}^3$ of the form $\bigl(\mathrm{vol}(K), 3\mathrm{W}_1(K), 3\mathrm{W}_2(K)\bigr)$ as K ranges over $\mathcal{K}^3$ or, equivalently, for a characterization of the set of points

$$\mathfrak{B} = \left\{ \left(x(K) = \frac{4\pi \mathrm{W}_1(K)}{3\mathrm{W}_2(K)^2}, y(K) = \frac{16\pi^2 \mathrm{vol}(K)}{9\mathrm{W}_2(K)^3} \right) \in [0,1]^2 : K \in \mathcal{K}^3 \right\}.$$

The latter set is called the *Blaschke diagram*, and the map $\mathcal{K}^3 \longrightarrow [0,1]^2$ given by $K \rightsquigarrow \bigl(x(K), y(K)\bigr)$, the *Blaschke map*. One of the main problems in this context is to describe the Blaschke diagram $\mathfrak{B}$. Thus, according to the known inequalities relating the quermassintegrals, namely, the Minkowski inequalities and the isoperimetric inequality for planar sets, the Blaschke diagram *contains* the shaded region in Figure 1, left.

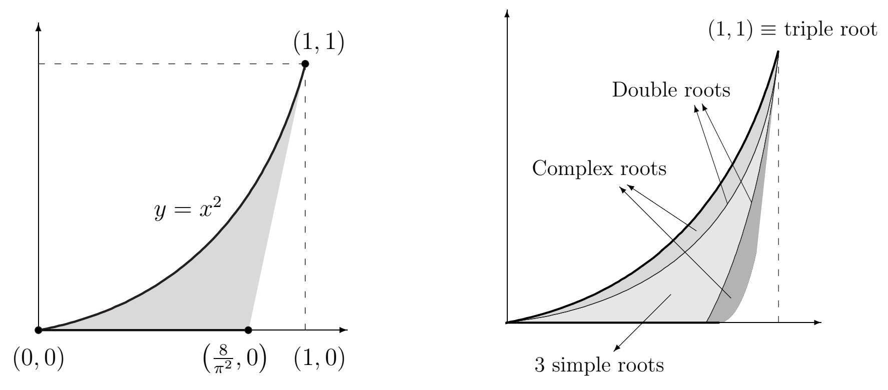

FIGURE 1. The Blaschke diagram and the type of roots of the Steiner polynomial.

Indeed, the Minkowski inequalities

$$\mathrm{W}_2(K)^2 \geq (4\pi/3)\mathrm{W}_1(K) \quad \text{and} \quad \mathrm{W}_2(K)^3 \geq (4\pi/3)^2 \operatorname{vol}(K)$$

(equality holds for balls in both inequalities) ensure that the diagram is a subset of the unit square $[0,1]^2$. Furthermore, the third Minkowski inequality

$$\mathrm{W}_1(K)^2 \geq \operatorname{vol}(K)\mathrm{W}_2(K)$$

corresponds to $y \leq x^2$, and since the cap-bodies are the extremal sets for this inequality, they are mapped to the points of the parabola $y = x^2$, from $(0,0)$ –the segments– to $(1,1)$ –the balls (see Figure 1, left). We recall that a cap-body is the convex hull of a ball and countably many points such that the line segment joining any pair of those points intersects the ball (the limit cases of a line segment and a ball are included in this definition).

Finally, all planar convex bodies in $\mathbb{R}^3$ satisfy the isoperimetric inequality

$$6\mathrm{W}_2(K)^2 \geq \pi^3\mathrm{W}_1(K) \quad \text{with} \quad \operatorname{vol}(K) = 0$$

(with equality for discs, cf. (2.1)); so, planar sets are mapped to the segment $[0, 8/\pi^2]$ of the x-axis –the point $(8/\pi^2, 0)$ corresponding to the disc D. Hence, if $8/\pi^2 < x \leq 1$, the y-coordinate has to be strictly positive, but the inequality satisfied by the convex bodies in this range is still unknown. The corresponding missing curve is known in the literature as *the missing boundary of the Blaschke diagram*. The problem of determining the known curve (and its higher dimensional version) remains open. Nowadays, there are two different conjectures, posed by Bieri and Sangwine-Yager (see [3, § 28] and [9] for a more detailed explanation).

Moreover, depending on the type of roots of its Steiner polynomial, a convex body is mapped into very precise regions/curves in the Blaschke diagram, as can be seen in Figure 1, right. These regions are determined by the equations $y = 3x - 2 \pm 2(1-x)^{3/2}$, and were obtained in [6].

2. Embedding 2- and 3-dimensional convex bodies in $\mathbb{R}^n$

In this section we consider k-dimensional convex bodies, $k = 2, 3$, embedded in a higher dimensional Euclidean space, and study the behavior of (the roots of) their Steiner polynomials. In the 3-dimensional case, the involved sets will follow a precise pattern when they are mapped into the Blaschke diagram.

If $K \in \mathcal{K}^n$ is a convex body contained in a k-dimensional subspace L, $k = 1, \ldots, n-1$, we write $\mathrm{W}_j^{(k)}(K)$, $j = 0, \ldots, k$, to denote the j-th quermassintegral of K (relative to the k-dimensional ball $B_k \subset L$) in dimension k. These k-dimensional quermassintegrals of K are related to the n-dimensional ones by means of the following identity:

$$\mathrm{W}_j(K) = \begin{cases} 0 & j = 0, \ldots, n-k-1, \\ \dfrac{\binom{k}{k+j-n}}{\binom{n}{j}} \dfrac{\kappa_j}{\kappa_{k+j-n}} \mathrm{W}_{k+j-n}^{(k)}(K) & j = n-k, \ldots, n. \end{cases} \tag{2.1}$$

Thus it is easy to check that the Steiner polynomial $f_{K,B_n}(z)$ is given by

$$f_{K,B_n}(z) = z^{n-k}\left[\sum_{i=0}^{k}\binom{k}{i}\frac{\kappa_{n-k+i}}{\kappa_i}\mathrm{W}_i^{(k)}(K)\,z^i\right]. \tag{2.2}$$

For a planar convex body K we denote, as usual, by $\mathrm{A}(K)=\mathrm{W}_0^{(2)}(K)$ and $\mathrm{p}(K)=2\mathrm{W}_1^{(2)}(K)$ its area and perimeter, respectively. Finally, we introduce the following notation: let

$$\mathcal{R}_{k,n}=\left\{K\in\mathcal{K}^k : \dim K = k,\ f_{K,B_n}(z) \text{ has only real roots}\right\}$$

where, in order to consider the Steiner polynomial $f_{K,B_n}(z)$, the set K is embedded in $\mathbb{R}^n$.

Theorem 2.1. *$\mathcal{R}_{2,n}\supset\mathcal{R}_{2,n+1}$ for all $n\geq 2$, and the inclusion is strict.*

Proof. Let $K\in\mathcal{K}^2$ with $\dim K=2$. Then by (2.2) we have

$$f_{K,B_n}(z) = z^{n-2}\left[\kappa_{n-2}\mathrm{A}(K)+\frac{\kappa_{n-1}}{2}\mathrm{p}(K)z+\kappa_n z^2\right],$$

being its roots 0 ($n-2$ times) and

$$\frac{-\kappa_{n-1}\mathrm{p}(K)\pm\sqrt{\kappa_{n-1}^2\mathrm{p}(K)^2-16\kappa_n\kappa_{n-2}\mathrm{A}(K)}}{4\kappa_n}.$$

Thus $f_{K,B_n}(z)$ will have only real roots if and only if

$$\mathrm{p}(K)^2\geq 16\frac{\kappa_n\kappa_{n-2}}{\kappa_{n-1}^2}\mathrm{A}(K). \tag{2.3}$$

Notice that for $n=2$ we get the classical isoperimetric inequality, and hence $\mathcal{R}_{2,2}=\mathcal{K}^2$. Since $\kappa_n\kappa_{n-2}/\kappa_{n-1}^2$ is a strictly increasing function in the dimension (see [5, Lemma 2.1]), we get that $\mathcal{R}_{2,n}\supset\mathcal{R}_{2,n+1}$ strictly. □

Remark 2.1. *The function $\kappa_n\kappa_{n-2}/\kappa_{n-1}^2$ is strictly increasing in the dimension and $\lim_{n\to\infty}\kappa_n\kappa_{n-2}/\kappa_{n-1}^2=1$ (see [5, Lemma 2.1]). Therefore, denoting by $\mathcal{R}_{2,\infty}$ the limit case, we have that*

$$\mathcal{R}_{2,\infty}=\left\{K\in\mathcal{K}^2 : \mathrm{p}(K)^2\geq 16\mathrm{A}(K)\right\},$$

and thus, the n-dimensional Steiner polynomial of any planar convex body verifying the inequality $\mathrm{p}(K)^2\geq 16\mathrm{A}(K)$ will have all its roots real for any value *of the dimension. In particular any square C_2 satisfies that property and has maximum area among all sets in $\mathcal{R}_{2,\infty}$ with fixed perimeter, because $\mathrm{p}(C_2)^2=16\mathrm{A}(C_2)$.*

Next we consider the case of 3-dimensional convex bodies.

Theorem 2.2. *$\mathcal{R}_{3,n}\supset\mathcal{R}_{3,n+2}$ for all $n\geq 3$, and the inclusion is strict. Moreover, $\mathcal{R}_{3,n}\supset\mathcal{R}_{3,n+1}$ does not hold.*

Proof. Let $K \in \mathcal{K}^3$ with $\dim K = 3$. Then by (2.2) we have

$$f_{K,B_n}(z) = z^{n-3}\left[\kappa_{n-3}\mathrm{W}_0^{(3)}(K) + \frac{3\kappa_{n-2}}{2}\mathrm{W}_1^{(3)}(K)z + \frac{3\kappa_{n-1}}{\pi}\mathrm{W}_2^{(3)}(K)z^2 + \kappa_n z^3\right].$$

In general one can check (see also [6, Section 2]) that a 3-dimensional polynomial of the form $a_0 + a_1 z + a_2 z^2 + z^3$ has three real roots if and only if

$$\text{(2.4)} \qquad \begin{cases} 2a_2^3 - 9(a_1 a_2 - 3a_0) + 2(a_2^2 - 3a_1)^{3/2} \geq 0, \\ 2a_2^3 - 9(a_1 a_2 - 3a_0) - 2(a_2^2 - 3a_1)^{3/2} \leq 0, \\ a_2^2 \geq 3a_1, \end{cases}$$

simultaneously. Or equivalently, when $a_2 \neq 0$, if and only if the inequalities

$$\text{(2.5)} \qquad y \geq 3x - 2 - 2(1-x)^{3/2}, \quad y \leq 3x - 2 + 2(1-x)^{3/2}, \quad x \leq 1$$

hold, where the coordinates (x, y) are given by $x = 3a_1/a_2^2$ and $y = 27a_0/a_2^3$.

Denoting by (x_n, y_n) the corresponding coordinates of the above polynomial $f_{K,B_n}(z)$, it is an easy computation to check that

$$\text{(2.6)} \qquad x_n = \frac{\kappa_n \kappa_{n-2} \kappa_2^2}{\kappa_{n-1}^2 \kappa_3 \kappa_1}\, x_3 =: h(n)\, x_3, \qquad y_n = \frac{\kappa_n^2 \kappa_{n-3} \kappa_2^3}{\kappa_{n-1}^3 \kappa_3^2}\, y_3 =: g(n)\, y_3,$$

and then, in order to prove that $\mathcal{R}_{3,n} \supset \mathcal{R}_{3,n+2}$ for all $n \geq 3$, we have to show that if the coordinates (x_{n+2}, y_{n+2}) satisfy (2.5), then (x_n, y_n) do so. Thus we assume that (x_{n+2}, y_{n+2}) satisfy (2.5). From $x_{n+2} = h(n+2)x_3 \leq 1$, i.e., $x_3 \leq 1/h(n+2)$, we get

$$x_n = h(n)x_3 \leq \frac{h(n)}{h(n+2)} = \frac{\kappa_{n+1}^2 \kappa_{n-2}}{\kappa_{n+2}\kappa_{n-1}^2},$$

and since for all $n \geq 2$

$$\text{(2.7)} \quad \frac{\kappa_{n+1}}{\kappa_{n-1}} = \frac{\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}+1\right)} \frac{\Gamma\left(\frac{n-1}{2}+1\right)}{\pi^{(n-1)/2}} = \frac{\pi\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+1}{2}+1\right)} = \frac{\pi\Gamma\left(\frac{n+1}{2}\right)}{\frac{n+1}{2}\Gamma\left(\frac{n+1}{2}\right)} = \frac{2\pi}{n+1},$$

we get

$$x_n \leq \frac{h(n)}{h(n+2)} = \left(\frac{2\pi}{n+1}\right)^2 \frac{\kappa_{n-2}\kappa_n}{\kappa_n \kappa_{n+2}} = \left(\frac{2\pi}{n+1}\right)^2 \frac{n}{2\pi}\frac{n+2}{2\pi} = \frac{n(n+2)}{(n+1)^2} < 1.$$

Finally we prove that (x_n, y_n) satisfies the first inequality in (2.5); for the second one the argument is analogous. Notice that, in this case, it is enough to consider $x \in [3/4, 1]$, since $y \geq 0$ always. From (2.6) and (2.7) we have

$$x_n = \frac{h(n)}{h(n+2)} x_{n+2} = \frac{n(n+2)}{(n+1)^2} x_{n+2},$$
$$y_n = \frac{g(n)}{g(n+2)} y_{n+2} = \frac{(n-1)(n+2)^2}{(n+1)^3} y_{n+2},$$

and since $y_{n+2} \geq 3x_{n+2} - 2 - 2(1 - x_{n+2})^{3/2}$, we get

$$y_n \geq \frac{g(n)}{g(n+2)}\left[3x_{n+2} - 2 - 2(1-x_{n+2})^{3/2}\right].$$

Thus it suffices to prove that

$$\begin{aligned}\frac{g(n)}{g(n+2)}\Big[3x_{n+2}-2-2(1-x_{n+2})^{3/2}\Big] &\geq 3x_n-2-2(1-x_n)^{3/2} \\ &= 3\frac{h(n)}{h(n+2)}x_{n+2}-2-2\left(1-\frac{h(n)}{h(n+2)}x_{n+2}\right)^{3/2}\end{aligned} \tag{2.8}$$

when $3/4 \leq x_{n+2} \leq 1$. If $x_{n+2} = 3/4$ one can easily check that (2.8) holds, since the left-hand side vanishes whereas the right one is strictly negative. Thus we can consider the function

$$F(x,h) = \frac{3hx-2-2\,(1-hx)^{3/2}}{3x-2-2(1-x)^{3/2}} \quad \text{for } (x,h) \in \left(\frac{3}{4},1\right] \times \left[\frac{15}{16},1\right].$$

Elementary computations show that $\big(d^2F/(dxdh)\big)(x,h) < 0$, which implies that $(dF/dx)(x,h) > (dF/dx)(x,1) = 0$, i.e., $F(x,h)$ is an increasing function in x, and thus

$$F(x,h) \leq F(1,h) = 3h-2-2(1-h)^{3/2}.$$

Therefore,

$$F\left(x_{n+2},\frac{h(n)}{h(n+2)}\right) \leq F\left(1,\frac{n(n+2)}{(n+1)^2}\right) = \frac{(n-1)(n+2)^2}{(n+1)^3} = \frac{g(n)}{g(n+2)},$$

which proves (2.8). Notice that since all the functions involved are strictly monotonous, there is equality just for $x_{n+2} = 1$.

We conclude the proof showing that the inclusion $\mathcal{R}_{3,n} \supset \mathcal{R}_{3,n+1}$ is not true. Indeed, for $n = 3$ we can find a convex body $K \in \mathcal{K}^3$ whose 3-dimensional Steiner polynomial has complex roots, but such that $f_{K,B_4}(z)$ has only real roots. The technique of the Blaschke diagram will be very useful for this purpose. For instance, let $\omega_0 = 11/(10\pi^2)$, $\omega_1 = 19/(15\pi)$ and $\omega_2 = 4/3$. If we prove the existence of $K \in \mathcal{K}^3$ with $\mathrm{vol}(K) = \omega_0$, $\mathrm{W}_1(K) = \omega_1$ and $\mathrm{W}_2(K) = \omega_2$, then the corresponding Steiner polynomials in dimensions 3 and 4 will be, respectively,

$$\begin{aligned} f_{K,B_3}(z) &= \frac{11}{10\pi^2} + \frac{19}{5\pi}z + 4z^2 + \frac{4\pi}{3}z^3 \quad \text{and} \\ f_{K,B_4}(z) &= \frac{11}{5\pi^2}z + \frac{19}{10}z^2 + \frac{16}{3}z^3 + \frac{\pi^2}{2}z^4, \end{aligned}$$

and an easy computation allows to check that the first one has complex roots, whereas all the roots of $f_{K,B_4}(z)$ are real (and simple). Thus, in order to conclude the proof, we just have to show the existence of $K \in \mathcal{K}^3$ satisfying the above conditions. So, for $\omega_0, \omega_1, \omega_2$, we consider the coordinates

$$x_\omega = \frac{4\pi\omega_1}{3\omega_2^2} = \frac{19}{20}, \qquad y_\omega = \frac{16\pi^2\omega_0}{9\omega_2^3} = \frac{33}{40}$$

and the corresponding image point (x_ω, y_ω) in the unit square $[0,1]^2$ (see Figure 2).

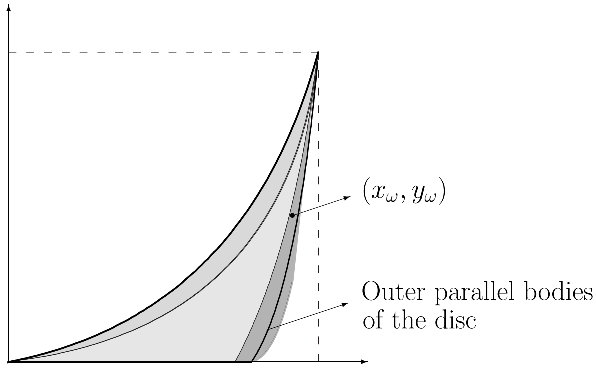

FIGURE 2. The point $(x_\omega, y_\omega) \in \mathfrak{B}$.

Since the boundary in the right-hand side of the diagram is unknown, we need to assure somehow that $(x_\omega, y_\omega) \in \mathfrak{B}$. In order to do it, we consider the unit disc D and the family $D + \lambda B_3$, $\lambda \geq 0$. Using (1.2) and (2.1), we get that the quermassintegrals of $D + \lambda B_3$ are given by

$$\begin{aligned}
\mathrm{vol}(D + \lambda B_3) &= 2\mathrm{A}(D)\lambda + \frac{\pi}{2}\mathrm{p}(D)\lambda^2 + \frac{4\pi}{3}\lambda^3 = 2\pi\lambda + \pi^2\lambda^2 + \frac{4\pi}{3}\lambda^3, \\
\mathrm{W}_1(D + \lambda B_3) &= \frac{2}{3}\mathrm{A}(D) + \frac{\pi}{3}\mathrm{p}(D)\lambda + \frac{4\pi}{3}\lambda^2 = \frac{2\pi}{3} + \frac{2\pi^2}{3}\lambda + \frac{4\pi}{3}\lambda^2, \\
\mathrm{W}_2(D + \lambda B_3) &= \frac{\pi}{6}\mathrm{p}(D) + \frac{4\pi}{3}\lambda = \frac{\pi^2}{3} + \frac{4\pi}{3}\lambda,
\end{aligned}$$

and thus, it is a direct computation to see that the family $D + \lambda B_3$, $\lambda \geq 0$, is mapped to the curve

$$y = 3x - 2 - 2\pi(12 - \pi^2)\left(\frac{1 - x}{\pi^2 - 8}\right)^{3/2} =: f(x).$$

Then one can easily check that $y_\omega > f(x_\omega)$, which implies that the point (x_ω, y_ω) lies in the part of the diagram bounded by the image curve $y = f(x)$ of the family $D + \lambda B_3$ (see Figure 2); hence, $(x_\omega, y_\omega) \in \mathfrak{B}$. This ensures the existence of $K \in \mathcal{K}^3$ such that $\mathrm{vol}(K) = \omega_0$, $\mathrm{W}_1(K) = \omega_1$ and $\mathrm{W}_2(K) = \omega_2$ (we even know that $K = M + \lambda_0 B_3$ for a convex body M with $\dim M = 2$ and a suitable $\lambda_0 > 0$, see [6, Corollary 3]). It concludes the proof. $\square$

Remark 2.2. *The functions $h(n)$, $g(n)$ (cf. (2.6)) are strictly increasing in the dimension, with $\lim_{n\to\infty} h(n) = 3\pi/8$ and $\lim_{n\to\infty} g(n) = 9\pi/16$. Thus, we denote by $\mathcal{R}_{3,\infty}$ the limit case, i.e.,*

$$\begin{aligned}
\mathcal{R}_{3,\infty} = \Bigg\{ K \in \mathcal{K}^3 : \frac{9\pi}{16}\, y_3 &\geq \frac{9\pi}{8}\, x_3 - 2 - 2\left(1 - \frac{3\pi}{8}x_3\right)^{3/2}, \\
\frac{9\pi}{16}\, y_3 &\leq \frac{9\pi}{8}\, x_3 - 2 + 2\left(1 - \frac{3\pi}{8}x_3\right)^{3/2}, \ x_3 \leq \frac{8}{3\pi} \Bigg\}.
\end{aligned}$$

An analogous argument to the one of the proof of Theorem 2.2 allows us to show that $\mathcal{R}_{3,\infty} \subset \mathcal{R}_{3,n}$ *for all* $n \geq 3$*, which means that the* n*-dimensional Steiner polynomial of any* 3*-dimensional convex body whose* (x_3, y_3)*-coordinates verify the above inequalities will have all its roots real for* any value *of the dimension. Notice that the* (x_3, y_3)*-coordinates are nothing but the* (x, y)*-coordinates defining the* Blaschke diagram*. Figure 3 shows the regions* $\mathcal{R}_{3,3}$ *and* $\mathcal{R}_{3,\infty}$*.*

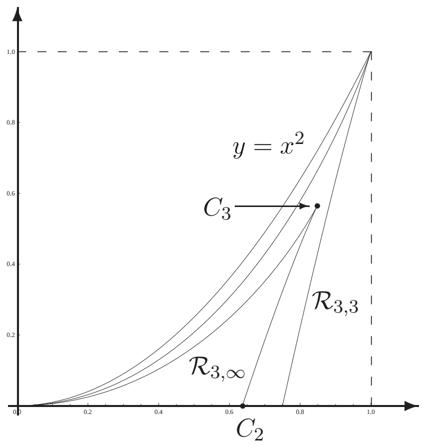

FIGURE 3. The regions $\mathcal{R}_{3,3}$ and $\mathcal{R}_{3,\infty}$.

It is easy to verify that the (x, y)*-coordinates of the orthogonal boxes with basis the unit square and height* $h \in [0, \infty]$ *are given by*

$$\left(\frac{8(2h+1)}{\pi(h+2)^2}, \frac{48h}{\pi(h+2)^3}\right),$$

and fill the boundary curve of $\mathcal{R}_{3,\infty}$*: when* $h \in [0, 1]$ *the right part of the boundary of* $\mathcal{R}_{3,\infty}$ *is obtained, from* $(2/\pi, 0)$ *(the square* C_2*) to the point* $\big(8/(3\pi), 16/(9\pi)\big)$ *(the cube* C_3*), see Figure 3; the left boundary curve is achieved for* $h \geq 1$*, being the limit case as* $h \to \infty$ *the point* $(0, 0)$ *(corresponding to the segments). In particular, (any dilation of) the cube* C_3 *has maximum volume among all sets in* $\mathcal{R}_{3,\infty}$ *with given quermassintegral* W_2*.*

3. ON THE DUAL BLASCHKE DIAGRAM

Next we move into the dual setting. The *dual Brunn-Minkowski theory* goes back to Lutwak [7, 8], and, among others, convex bodies are replaced by star bodies, the Minkowski sum by the radial addition and the support function by the radial function (see e.g. [10, Section 9.3]). If $x, y \in \mathbb{R}^n$, the radial addition $x \tilde{+} y$ is defined as

$$x \tilde{+} y = \begin{cases} x + y & \text{if } x, y \text{ are linearly dependent,} \\ 0 & \text{otherwise.} \end{cases}$$

In general, the radial sum $K\widetilde{+}E = \{x\widetilde{+}y : x \in K,\, y \in E\}$ of two convex bodies K, E is not a convex set, but the radial sum of two star bodies is again a star body. In order to define star bodies, we call a non-empty set $S \subseteq \mathbb{R}^n$ *starshaped* (with respect to the origin) if the segment $[0, x] \subseteq S$ for all $x \in S$. For a compact starshaped set K its *radial function* $\rho_K : \mathbb{S}^{n-1} \longrightarrow \mathbb{R}_{\geq 0}$ is defined by $\rho_K(u) = \max\{\rho \geq 0 : \rho u \in K\}$. If this function is positive and continuous then K is called a *star body*. In particular, any star body has non-empty interior and any convex body containing the origin in its interior is a star body. We denote by $\mathcal{S}_0^n$ the set of all star bodies in $\mathbb{R}^n$.

It is easy to see that, in this case, the volume of the radial sum $K\widetilde{+}\lambda E$, $K, E \in \mathcal{S}_0^n$, is also expressed as a polynomial of degree n in λ (see e.g. [10, page 508]), the so-called (relative) *dual Steiner formula*, and it is written as

$$\mathrm{vol}\big(K\widetilde{+}\lambda E\big) = \sum_{i=0}^{n} \binom{n}{i} \widetilde{\mathrm{W}}_i(K; E)\lambda^i. \tag{3.1}$$

The coefficients $\widetilde{\mathrm{W}}_i(K; E)$ are the *dual quermassintegrals* of K and E, and they are special cases of the dual mixed volumes, which were introduced for the first time by Lutwak in [7]. Dual quermassintegrals satisfy that $\widetilde{\mathrm{W}}_0(K; E) = \mathrm{vol}(K)$, $\widetilde{\mathrm{W}}_n(K; E) = \mathrm{vol}(E)$ and $\widetilde{\mathrm{W}}_i(K; E) = \widetilde{\mathrm{W}}_{n-i}(E; K)$; furthermore, they are homogeneous of degree $n - i$ (respectively, degree i) in the first (respectively, second) argument. When $E = B_n$, we write for short $\widetilde{\mathrm{W}}_i(K) = \widetilde{\mathrm{W}}_i(K; B_n)$.

It is well-known that for $K, E \in \mathcal{S}_0^n$,

$$\widetilde{\mathrm{W}}_j(K; E)^{k-i} \leq \widetilde{\mathrm{W}}_i(K; E)^{k-j}\widetilde{\mathrm{W}}_k(K; E)^{j-i}, \quad i < j < k, \tag{3.2}$$

which are the "dual" counterpart to the classical Aleksandrov-Fenchel inequalities (see e.g. [10, (9.40)]). Now equality holds if and only if K and E are dilates. In dimension $n = 3$ and when $E = B_3$, (3.2) translates into

$$\widetilde{\mathrm{W}}_1(K)^2 \leq \mathrm{vol}(K)\widetilde{\mathrm{W}}_2(K), \tag{3.3}$$

$$\widetilde{\mathrm{W}}_2(K)^2 \leq \kappa_3\widetilde{\mathrm{W}}_1(K), \tag{3.4}$$

$$\widetilde{\mathrm{W}}_1(K)^3 \leq \kappa_3\mathrm{vol}(K)^2, \tag{3.5}$$

$$\widetilde{\mathrm{W}}_2(K)^3 \leq \kappa_3^2\mathrm{vol}(K). \tag{3.6}$$

Next we follow the Blaschke idea and define the *dual Blaschke map* as

$$\begin{aligned} &\mathcal{S}_0^3 \longrightarrow [0, 1]^2 \\ &K \rightsquigarrow \big(x(K), y(K)\big) = \left(\frac{\widetilde{\mathrm{W}}_1(K)}{\big(\kappa_3\mathrm{vol}(K)^2\big)^{1/3}}, \frac{\widetilde{\mathrm{W}}_2(K)}{\big(\kappa_3^2\mathrm{vol}(K)\big)^{1/3}}\right), \end{aligned}$$

which is well-defined because (3.5) and (3.6) ensure that both $x(K), y(K) \in (0, 1]$. We call the image $\widetilde{\mathfrak{B}}$ of the above map *the dual Blaschke diagram*, and the question arises to determine the set $\widetilde{\mathfrak{B}}$.

First we observe that this choice of the coordinates allows us to assure that all the sets with the "same shape" are mapped to the same point in the diagram because, due to the homogeneity of the dual quermassintegrals, the functionals $\widetilde{\mathrm{W}}_1(K)/\mathrm{vol}(K)^{2/3}$ and $\widetilde{\mathrm{W}}_2(K)/\mathrm{vol}(K)^{1/3}$ are invariant under dilations.

On the other hand, (3.3) and (3.4) correspond to $y \geq x^2$ and $y^2 \leq x$, respectively, and therefore, $\widetilde{\mathfrak{B}}$ will be contained in the region determined by these two curves (see Figure 4). But since the only extremal sets for these inequalities are balls, the boundary cannot be considered, except the point $(1,1)$ (where B_3 is mapped). As we will see next, this fact will provide a first structural difference between the classical and the dual Blaschke diagrams: the first one is closed, but not the dual one.

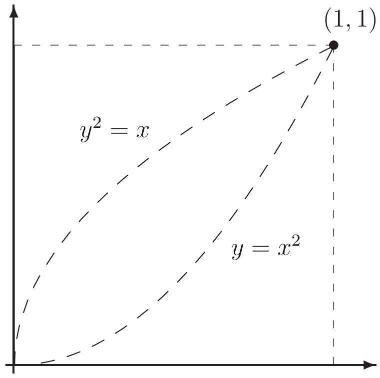

FIGURE 4. The dual Blaschke diagram.

Remark 3.1. *We observe that contrary to the classical case, here we can fix any star body E (not only the ball) and construct the dual Blaschke map using the more general dual quermassintegrals $\widetilde{\mathrm{W}}_i(K;E)$. In the classical Blaschke diagram the ball plays a crucial role, since it allows us to consider planar sets and relate the area and perimeter with their 3-dimensional quermassintegrals.*

Theorem 3.1. *The dual Blaschke diagram $\widetilde{\mathfrak{B}}$ is the set*

$$\widetilde{\mathfrak{B}} = \big\{(x,y) \in [0,1]^2 : x^2 < y < \sqrt{x}\big\} \cup \big\{(1,1)\big\}.$$

In particular, $\widetilde{\mathfrak{B}}$ is simply connected and not closed.

Proof. In [1, Theorem 2.2] a characterization of dual quermassintegrals is proved in terms of the positive definiteness of particular matrices. This result, together with [1, Lemma 2.2], reads as follows when $n = 3$: given $\omega_i > 0$, $i = 0, \ldots, 3$, there exists a star body $K \in \mathcal{S}_0^3$ with

$$\widetilde{\mathrm{W}}_i\left(\left(\frac{\omega_0}{\kappa_3}\right)^{1/3} B_3; K\right) = \omega_i \quad \text{for all} \;\; i = 0, \ldots, 3$$

if and only if, either there exist $0 < a < b$ such that the Hankel matrices

$$\begin{pmatrix} \omega_1 - a\omega_0 & \omega_2 - a\omega_1 \\ \omega_2 - a\omega_1 & \omega_3 - a\omega_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} b\omega_0 - \omega_1 & b\omega_1 - \omega_2 \\ b\omega_1 - \omega_2 & b\omega_2 - \omega_3 \end{pmatrix}$$

are positive definite, or $\omega_i = \lambda^i \omega_0$ for some $\lambda > 0$ and $i = 1, 2, 3$, and in this case $K = \lambda B_3$.

On one hand we observe that, due to the homogeneity and the symmetry of the dual quermassintegrals, the condition $\widetilde{\mathrm{W}}_i\big((\omega_0/\kappa_3)^{1/3}B_3; K\big) = \omega_i$ can be written as

$$\widetilde{\mathrm{W}}_{3-i}(K) = \left(\frac{\kappa_3}{\omega_0}\right)^{(3-i)/3} \omega_i;$$

therefore, for the sake of simplicity we can always assume that the first value $\omega_0 = \kappa_3$. On the other hand, we recall that a matrix is positive definite if and only if its leading principal minors are all positive.

Thus, altogether, we get that the above characterization can be read as follows: given $\omega_i > 0$, $i = 1, 2, 3$, there exists a star body $K \in \mathcal{S}^3_0$, $K \neq \lambda B_3$ for all $\lambda > 0$, with $\widetilde{\mathrm{W}}_{3-i}(K) = \omega_i$ for every $i = 1, 2, 3$, if and only if there exist $0 < a < b$ such that

$$\begin{aligned} a &< \frac{\omega_1}{\kappa_3} \quad \text{and} \quad \omega_1\omega_3 - \omega_2^2 + a^2(\kappa_3\omega_2 - \omega_1^2) - a(\kappa_3\omega_3 - \omega_1\omega_2) > 0, \\ b &> \frac{\omega_1}{\kappa_3} \quad \text{and} \quad \omega_1\omega_3 - \omega_2^2 + b^2(\kappa_3\omega_2 - \omega_1^2) - b(\kappa_3\omega_3 - \omega_1\omega_2) > 0. \end{aligned} \tag{3.7}$$

Clearly $\widetilde{\mathfrak{B}} \subset \big\{(x, y) \in [0, 1]^2 : x^2 < y < \sqrt{x}\big\} \cup \big\{(1, 1)\big\}$, and so we have to prove the reverse inclusion.

Obviously $(1, 1) \in \widetilde{\mathfrak{B}}$ because B_3 is mapped to the point $(1, 1)$. So we take a point $(x_0, y_0) \in [0, 1]^2$ satisfying the condition $x_0^2 < y_0 < \sqrt{x_0}$ and we have to show the existence of a star body $K \in \mathcal{S}^3_0$ whose image via the dual Blaschke map is the point (x_0, y_0). In order to do it we write $x_0 = y_0^2 + \varepsilon$, where ε may range, at most, in the open interval

$$\varepsilon \in \big(0, \sqrt{y_0} - y_0^2\big)\,; \tag{3.8}$$

indeed, notice that the possible end-point $\big(y_0^2 + \sqrt{y_0} - y_0^2, y_0\big) = \big(\sqrt{y_0}, y_0\big)$ would lie on the right boundary curve $y = x^2$.

Next we consider the triple of numbers

$$\omega_1 = \sqrt{\kappa_3}\, y_0, \qquad \omega_2 = y_0^2 + \varepsilon, \qquad \omega_3 = \frac{1}{\sqrt{\kappa_3}}. \tag{3.9}$$

If we show that there exist $0 < a < b$ such that (3.7) holds for these values, then we can guarantee the existence of $K \in \mathcal{S}^3_0$ with

$$\mathrm{vol}(K) = \frac{1}{\sqrt{\kappa_3}}, \quad \widetilde{\mathrm{W}}_1(K) = y_0^2 + \varepsilon, \quad \widetilde{\mathrm{W}}_2(K) = \sqrt{\kappa_3}\, y_0$$

and, of course, $\widetilde{\mathrm{W}}_3(K) = \kappa_3$, and hence

$$\bigl(x(K), y(K)\bigr) = \left(\frac{\widetilde{\mathrm{W}}_1(K)}{\bigl(\kappa_3 \mathrm{vol}(K)^2\bigr)^{1/3}}, \frac{\widetilde{\mathrm{W}}_2(K)}{\bigl(\kappa_3^2 \mathrm{vol}(K)\bigr)^{1/3}} \right) = (y_0^2 + \varepsilon, y_0),$$

i.e., $(x_0, y_0) \in \widetilde{\mathfrak{B}}$, as required.

Thus, substituting (3.9) in (3.7), we have to show that there exist $a < b$ such that

$$0 < a < \frac{y_0}{\sqrt{\kappa_3}} < b, \tag{3.10}$$

$$y_0 - (y_0^2 + \varepsilon)^2 + \kappa_3 a^2 \varepsilon - \sqrt{\kappa_3}\, a\bigl(1 - y_0(y_0^2 + \varepsilon)\bigr) > 0, \tag{3.11}$$

and

$$y_0 - (y_0^2 + \varepsilon)^2 + \kappa_3 b^2 \varepsilon - \sqrt{\kappa_3}\, b\bigl(1 - y_0(y_0^2 + \varepsilon)\bigr) > 0. \tag{3.12}$$

Denoting by a_+, a_-, b_+, b_- the corresponding roots of the 2nd-degree polynomials (in a and b) in, respectively, (3.11) and (3.12), namely,

$$a_\pm, b_\pm = \frac{\sqrt{3}}{4\sqrt{\pi\varepsilon}} \left[1 - y_0^3 - \varepsilon y_0 \pm \sqrt{y_0^6 + 6\varepsilon y_0^4 - 2y_0^3 + 9\varepsilon^2 y_0^2 - 6\varepsilon y_0 + 4\varepsilon^3 + 1} \right]$$

we get that (3.11) and (3.12) hold if and only if, either $a < a_-$ or $a > a_+$, and either $b < b_-$ or $b > b_+$.

It is easy to see, on one hand, that $b_+ > y_0/\sqrt{\kappa_3}$ for all $\varepsilon > 0$, and hence we can guarantee that (3.12) holds for any $b > b_+$.

On the other hand, a direct computation proves that $a_- > 0$ if and only if $0 < \varepsilon < \sqrt{y_0} - y_0^2$ (cf. (3.8)). Therefore, and altogether, when $\varepsilon \in \bigl(0, \sqrt{y_0} - y_0^2\bigr)$, it suffices to take $b > b_+$ and $a < \min\bigl\{a_-, y_0/\sqrt{\kappa_3}\bigr\}$ to assure the validity of (3.10), (3.11) and (3.12).

We finally notice that the previous argument is valid when ε ranges over the above interval, and therefore, for a fixed height y_0 in the diagram, the *full open* segment

$$\Bigl((y_0^2, y_0), \bigl(y_0^2 + \sqrt{y_0} - y_0^2, y_0\bigr) \Bigr) = \Bigl((y_0^2, y_0), \bigl(\sqrt{y_0}, y_0\bigr) \Bigr) \subset \widetilde{\mathfrak{B}}.$$

It concludes the proof. □

Theorem 3.1 constitutes a second significative difference between the classical and the dual Blaschke problems: in the second case, the dual Aleksandrov-Fenchel inequalities completely determine the dual Blaschke diagram; the classical Aleksandrov-Fenchel inequalities, however, are not enough to determine the Blaschke diagram, since at least one inequality is unknown (the missing boundary of the Blaschke diagram).

We observe that in [1, Proof of Proposition 4.2] it was proved that the dual Aleksandrov-Fenchel inequality $\widetilde{\mathrm{W}}_1(K)^2 \le \kappa_2 \widetilde{\mathrm{W}}_0(K)$ characterizes the dual quermassintegrals when $n = 2$. As an immediate consequence of Theorem 3.1 we get the following new characterization of dual quermassintegrals in dimension $n = 3$.

Corollary 3.1. *Let $\omega_i > 0$, $i = 0, 1, 2$, be positive real numbers. There exists a star body $K \in \mathcal{S}_0^3$ such that $\widetilde{\mathrm{W}}_i(K) = \omega_i$, $i = 0, 1, 2$, if and only if either they verify the dual Aleksandrov-Fenchel inequalities* (3.3)–(3.6) strictly*, or $\omega_i = \lambda^{3-i}\kappa_3$ for some $\lambda > 0$ and $i = 0, 1, 2$, and in this case $K = \lambda B_3$.*

Proof. Obviously, if the numbers ω_i are dual quermassintegrals, then they verify the dual Aleksandrov-Fenchel inequalities (3.3)–(3.6), either strictly, or with equality in all of them; in this last case, $K = \lambda B_3$ for some $\lambda > 0$.

In order to prove the converse we assume that K is not a ball, and thus the point

$$(x_0, y_0) = \left(\frac{\omega_1}{\left(\kappa_3\omega_0^2\right)^{1/3}}, \frac{\omega_2}{\left(\kappa_3^2\omega_0\right)^{1/3}}\right) \in \left\{(x, y) \in [0, 1]^2 : x^2 < y < \sqrt{x}\right\}$$

because the numbers $\omega_0, \omega_1, \omega_2$ satisfy the (strict) inequalities (3.3)–(3.6). Then, Theorem 3.1 ensures that $(x_0, y_0) \in \widetilde{\mathfrak{B}}$, and hence, there exists a star body $\bar{K} \in \mathcal{S}_0^3$ such that $x_0 = x\big(\bar{K}\big)$ and $y_0 = y\big(\bar{K}\big)$, i.e.,

$$\frac{\omega_1}{\omega_0^{2/3}} = \frac{\widetilde{\mathrm{W}}_1\big(\bar{K}\big)}{\mathrm{vol}\big(\bar{K}\big)^{2/3}} \quad \text{and} \quad \frac{\omega_2}{\omega_0^{1/3}} = \frac{\widetilde{\mathrm{W}}_2\big(\bar{K}\big)}{\mathrm{vol}\big(\bar{K}\big)^{1/3}}.$$

Taking $K = \big(\omega_0/\mathrm{vol}(\bar{K})\big)^{1/3}\bar{K} \in \mathcal{S}_0^3$, we get $\mathrm{vol}(K) = \omega_0$, $\widetilde{\mathrm{W}}_1(K) = \omega_1$ and $\widetilde{\mathrm{W}}_2(K) = \omega_2$, as required. □

Regarding the study of the roots of the dual Steiner polynomial, it has been recently developed in [1]. In particular, if we look for the different types of roots that a (n-dimensional) dual Steiner polynomial can have, in [1, Proposition 4.4] it was proved that all its roots are real if and only if $K = \lambda E$ for some $\lambda > 0$, i.e., the only possibility is that all the roots are equal. Therefore, in dimension 3 we can only have either a real root and two (conjugate) complex ones (and so all these sets are mapped into the interior of the diagram), or three equal real roots (corresponding to the point $(1, 1)$).

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